Annotated Bibliography

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- [Bai07] We calculate the Euler characteristics of all of the Teichmuller curves in the moduli space $\mathcal{M}_2$ of genus two Riemann surfaces which are generated by holomorphic one-forms with a single double zero, which were discovered and classified in [Cal04] and [McM03]. These curves can all be embedded in Hilbert modular surfaces and our main result is that the Euler characteristic of a Teichmuller curve is proportional to the Euler characteristic of the Hilbert modular surface on which it lies.

  This is done by defining Hilbert modular forms which vanish along these Teichmuller curves. To correct for errors arising from the behavior of these forms at infinity, we construct smooth compactifications of Hilbert modular surfaces such that their embeddings in $\mathcal{M}_2$ extend to embeddings of the compactification in the Deligne-Mumford compactification, and moreover the Hilbert modular forms extend holomorphically over the compactification.

  We apply these results to calculate the Siegel-Veech constants for counting closed billiards paths in certain L-shaped polygons. We also calculate the Lyapunov exponents of the Kontsevich-Zorich cocycle for any ergodic $\mathrm{SL}_2\mathbb{R}$-invariant measure on the moduli space of Abelian differentials in genus two. At the time, these Lyapunov exponents were calculated in unpublished work of Kontsevich and Zorich, which later appeared in [EKZ14].

- [Bai10] We compute the volumes of the eigenform loci in the moduli space of genus two Abelian differentials. This is done by expressing these volumes as intersection numbers of cohomology classes on the compactifications of Hilbert modular surfaces in [Bai07]. From this, we obtain asymptotic formulas for counting closed billiards paths in certain L-shaped polygons with barriers.

Note, the last section has a gap which was revealed in an erratum. This was corrected in my subsequent work [BSW16].

- [BM12] In the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces, consider the locus $\mathcal{RM}_\mathcal{O}$ of Riemann surfaces whose Jacobians have real multiplication by the order $\mathcal{O}$ in a totally real number field F of degree g. If $g = 2$ or 3, we compute the closure of $\mathcal{RM}_\mathcal{O}$ in the Deligne-Mumford compactification of $\mathcal{M}_g$ and the closure of the locus of eigenforms over $\mathcal{RM}_\mathcal{O}$ in the Deligne-Mumford compactification of the moduli space of holomorphic one-forms. For higher genera, we give strong necessary conditions for a stable curve to be in the boundary of $\mathcal{RM}_\mathcal{O}$. Boundary strata of $\mathcal{RM}_\mathcal{O}$ are parameterized by configurations of elements of the field F satisfying a strong geometry of numbers type restriction.

  We apply this computation to give evidence for the conjecture that there are only finitely many algebraically primitive Teichmueller curves in $\mathcal{M}_3$. In particular, we prove that there are only finitely many algebraically primitive Teichmueller curves generated by a one-form having two zeros of order 3 and 1. We also present the results of a computer search for algebraically primitive Teichmüller curves generated by a one-form having a single zero. As another application, we show that the eigenform bundle over $\mathcal{RM}_\mathcal{O}$ is never $\mathrm{SL}_2\mathbb{R}$-invariant when $\mathcal{O}$ is a maximal order.

- [BM14] We prove that the generic point of a Hilbert modular four-fold is not a Jacobian. The proof applies the restrictions of the closure of the real multiplication locus in the Deligne-Mumford compactification from [BM12]. This completes the main result of [dJZ07], which establishes the analogous result for larger dimensions using very different techniques.

- [BHM16] We prove that the moduli space of compact genus three Riemann surfaces contains only finitely many algebraically primitive Teichmüller curves.

  For the stratum consisting of holomorphic one-forms with a single zero, our approach to finiteness uses the Harder-Narasimhan filtration of the Hodge bundle over a Teichmueller curve to obtain new information on the locations of the zeros of eigenforms. By passing to the boundary of moduli space, this gives explicit constraints on the cusps of Teichmüller curves in terms of cross-ratios of six points on a projective line. These constraints are akin to those that appear in Zilber and Pink's conjectures on unlikely intersections in diophantine geometry. However, in our case one is lead naturally to the intersection of a surface with a family of codimension two algebraic subgroups of $\mathbb{G}_m^n \times \mathbb{G}_a^n$ (rather than the more standard $\mathbb{G}_m^n$). The ambient algebraic group lies outside the scope of Zilber's Conjecture but we are nonetheless able to prove a sufficiently strong height bound.

For the generic stratum in genus three, we obtain global torsion order bounds through a computer search for subtori of a codimension-two subvariety of $\mathbb{G}_m^9$. These torsion bounds together with new bounds for the moduli of horizontal cylinders in terms of torsion orders yields finiteness in this stratum. The intermediate strata are handled with a mix of these techniques.

- [BSW16] We study dynamics of the horocycle flow on strata of translation surfaces, introduce new invariants for ergodic measures, and analyze the interaction of the horocycle flow and real Rel surgeries. We use this analysis to complete and extend results of Calta and Wortman [CW] classifying horocycle-invariant measures in the eigenform loci. We classify the orbit-closures and prove that every orbit is equidistributed in its orbit-closure. We also prove equidistribution statements regarding limits of sequences of measures, some of which have applications to counting problems.

- [BCG$^+$18] We describe the closure of the strata of abelian differentials with prescribed type of zeros and poles, in the projectivized Hodge bundle over the Deligne-Mumford moduli space of stable curves with marked points. We provide an explicit characterization of pointed stable differentials in the boundary of the closure, both a complex analytic proof and a flat geometric proof for smoothing the boundary differentials, and numerous examples. The main new ingredient in our description is a global residue condition arising from a full order on the dual graph of a stable curve.

  The compactification constructed in this paper was later discovered to be highly singular. This defect was rectified in our subsequent work [BCG$^+$].

- [BCG$^+$19] This paper establishes results analogous to [BCG$^+$18] for moduli spaces of k-differentials on Riemann surfaces. A k-differential on a Riemann surface is a section of the k-th power of the canonical line bundle. Loci of k-differentials with prescribed number and multiplicities of zeros and poles form a natural stratification of the moduli space of k-differentials. In this paper we give a complete description for the compactification of the strata of k-differentials in terms of pointed stable k-differentials, for all k. The upshot is a global k-residue condition that can also be reformulated in terms of admissible covers of stable curves. Moreover, we study properties of k-differentials regarding their deformations, residues, and flat geometric structure.

- [BCG$^+$] We construct a compactification of the moduli spaces of abelian differentials on Riemann surfaces with prescribed zeroes and poles. This compactification, called the moduli space of multi-scale differentials, is a complex orbifold with normal crossing boundary. Locally, our compactification can be described as the normalization of an explicit blowup of the incidence variety compactification, which was studied in [BCG$^+$18] as the closure of the stratum of abelian differentials in the closure of the Hodge bundle. We also define families of projectivized multi-scale differentials, which gives a proper Deligne-Mumford stack, and our compactification is the orbifold corresponding to it. Moreover, we perform a real oriented blowup of the unprojectivized moduli space of multi-scale differentials such that the $\mathrm{GL}_2\mathbb{R}$-action in the interior of the moduli space extends continuously to the boundary.

- [BJJP20] Let S be a closed topological surface. Haupt's theorem (see [Hau20] or [Kap]) provides necessary and sufficient conditions for a complex-valued character of the first integer homology group of S to be realized by integration against a complex-valued 1-form that is holomorphic with respect to some complex structure on S. We prove a refinement of this theorem that takes into account the divisor data of the 1-form. The proof relies heavily on the recent classification of closures of leaves of the isoperiodic foliation in the moduli space of holomorphic one-forms from [CDF].

- [ABW22] We construct moduli spaces of complex affine and dilation surfaces. Using ideas of Veech [Vee93], we show that the the moduli space of affine surfaces with fixed genus and with cone points of fixed complex order is a holomorphic affine bundle over the moduli space of Riemann surfaces. Similarly, the moduli space of dilation surfaces is a covering space of the moduli space of Riemann surfaces.

  As an application, we classify the connected components of the moduli space of dilation surfaces and show that any component is an orbifold $K(G, 1)$ where G is the framed mapping class group of Calderon-Salter [CS20].

References

[ABW22] P. Apisa, M. Bainbridge, and J. Wang. Moduli spaces of complex affine and dilation surfaces, 2022, arxiv.2204.05257.

[Bai07] M. Bainbridge. Euler characteristics of Teichmüller curves in genus two. *Geom. Topol.*, 11:1887–2073, 2007.

[Bai10] M Bainbridge. Billiards in L-shaped tables with barriers. *Geom. Funct. Anal.*, 20(2):299–356, 2010.

[BCG$^+$] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller. The moduli space of multi-scale differentials, arXiv:1910.13492.

[BCG$^+$18] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller. Compactification of strata of Abelian differentials. *Duke Math. J.*, 167(12):2347–2416, 2018.

[BCG$^+$19] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller. Strata of k-differentials. *Algebr. Geom.*, 6(2):196–233, 2019.

[BHM16] M. Bainbridge, P. Habegger, and M. Möller. Teichmüller curves in genus three and just likely intersections in $\mathbf{G}_m^n \times \mathbf{G}_a^n$. *Publ. Math. Inst. Hautes Études Sci.*, 124:1–98, 2016.

[BJJP20] M. Bainbridge, C. Johnson, C. Judge, and I. Park. Haupt's theorem for strata of abelian differentials, 2020, arXiv:2002.12901.

[BM12] M. Bainbridge and M. Möller. The Deligne-Mumford compactification of the real multiplication locus and Teichmüller curves in genus 3. *Acta Math.*, 208(1):1–92, 2012.

[BM14] M. Bainbridge and M. Möller. The locus of real multiplication and the Schottky locus. *J. Reine Angew. Math.*, 686:167–186, 2014.

[BSW16] M. Bainbridge, J. Smillie, and B. Weiss. Horocycle dynamics: new invariants and eigenform loci in the stratum $\mathcal{H}(1,1)$, 2016, arxiv.1603.00808, to appear in *Memoirs of the AMS*.

[Cal04] K. Calta. Veech surfaces and complete periodicity in genus two. *J. Amer. Math. Soc.*, 17(4):871–908, 2004.

[CDF] G. Calsamiglia, B. Deroin, and S. Francaviglia. A transfer principle: from periods to isoperiodic foliations, arxiv.1511.07635.

[CS20] A. Calderon and N. Salter. Framed mapping class groups and the monodromy of strata of abelian differentials, 2020, arXiv:2002.02472.

[CW] K. Calta and K. Wortman. On unipotent flows in H(1,1), arXiv:math.DS/0702238.

[dJZ07] J. de Jong and S.-W. Zhang. Generic abelian varieties with real multiplication are not Jacobians. In *Diophantine geometry*, volume 4 of *CRM Series*, pages 165–172. Ed. Norm., Pisa, 2007.

[EKZ14] A. Eskin, M. Kontsevich, and A. Zorich. Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow. *Publ. Math. Inst. Hautes Études Sci.*, 120:207–333, 2014.

[Hau20] O. Haupt. Ein Satz über die Abelschen Integrale 1. Gattung. *Math. Z.*, 6(3-4):219–237, 1920.

[Kap] M. Kapovich. Periods of abelian differentials and dynamics, arxiv.1710.11234.

[McM03] C. McMullen. Billiards and Teichmüller curves on Hilbert modular surfaces. *J. Amer. Math. Soc.*, 16(4):857–885, 2003.

[Vee93] W. Veech. Flat surfaces. *Amer. J. Math.*, 115(3):589–689, 1993.